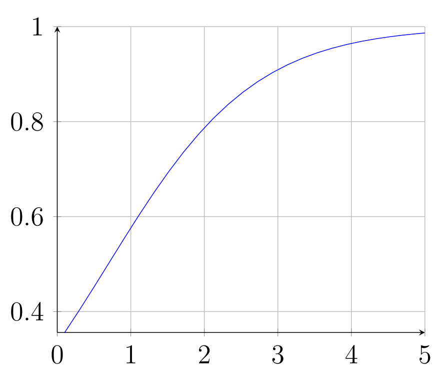
Math 208 I, Midterm 1 Name: ______________________

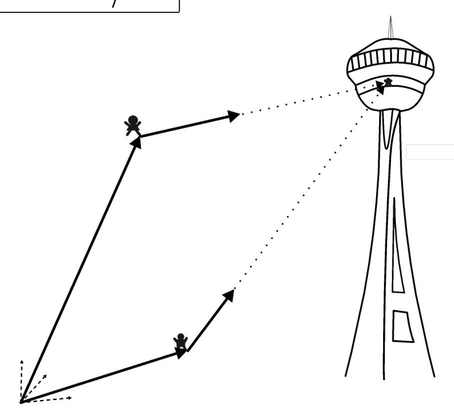
Signature: ______________________________________

Student ID #: ______________ Section #: ______________

- You are allowed a Ti-30x IIS Calculator and one 8.5×11 inch paper with handwritten notes on both sides. Other calculators, electronic devices (e.g. cell phones, laptops, etc.), notes, and books are **not** allowed.
- Some questions require you to explain answers: no explanation, no credit.
- Try to show your work on all questions: no work, no partial credit.
- You may use the back of the exam for scratch work: please submit any additional paper you use.
- Place a box around your answer to each question.
- Raise your hand if you have a question.

1	/10
2	/10
3	/10
4	/10
5	/10
T	/50





Good Luck!

(1) A logistic curve is a curve in the (x, y)-plane defined by an equation of the form $y(1 + ae^{-x}) = b$. (See coverpage for an illustration.)

(a) (4pts) Write a system of linear equations in a, b that can be used to fit a logistic curve to the following values of (x, y): $(0, 1/3), (\ln 2, 1/2)$.

(No need to simplify...yet.) $\begin{cases} e^{-0}/3a + 1/3 = b \\ e^{-\ln 2}/2a + 1/2 = b \end{cases}$, or in a standard, simplified form, $\begin{cases} -a + 3b = 1 \\ -a + 4b = 2 \end{cases}$

(b) (4pts) Solve this system (Hint: recall $e^0 = 1$, and $e^{-\ln x} = 1/x$.) From the RREF

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

we see that there is a unique solution, $a = 2$ and $b = 1$.

(c) (2pts) How many equations can we add to this system without violating the existence of a solution? Explain. As many as we want. There are infinitely many different lines in the plane passing through the point $(2, 1)$, and each one is defined by a linear equation different from the rest.

(2) (a) (7pts) Determine a 2×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ in *reduced echelon form*, such that z is a free variable and such that

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is the general solution to the system

$$\begin{aligned} ax + by + cz &= -1 \\ dx + ey + fz &= 2. \end{aligned}$$

The general form of the system and its solutions $\mathbf{x}$ suggests that the third column of A should be free, and that the pivot variables x_1, x_2 satisfy $x_1 - x_3 = -1$, and $x_2 - x_3 = 2$. Thus, if we set

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

we see that A is in RREF, and the inhomogeneous system in question has the desired form.

(b) (3pts) Consider the linear transformation associated to this matrix:

$$T_A(x, y, z) = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \end{pmatrix}$$

Calculate $T_A(1, 1, 1)$. $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(3) In each case below describe all values of t (when possible) for which the given vectors are linearly **dependent**. (2.5 pts each)

(a) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}\pi\\4\end{pmatrix}, \begin{pmatrix}\sqrt{2}\\2024\end{pmatrix}, \begin{pmatrix}t\\7\end{pmatrix}$ All t—more than n vectors in $\mathbb{R}^n$ are always dependent.

(b) $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\t\\5\end{pmatrix}$ No t—the first and second vector are never multiples.

(c) $\begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\1+t\end{pmatrix}, \begin{pmatrix}1\\t^2-3\\\cos(t)\end{pmatrix}$ Only $t = 1$, in which case the first two vectors are multiples of each other.

(d) $\begin{pmatrix}0\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}t\\t\\2\\0\end{pmatrix}$ Call the vectors v_1, v_2, v_3, and suppose we had a linear dependence $c_1v_1 + c_2v_2 + c_3v_3 = 0$ specified by scalars $c_1, c_2, c_3 \in \mathbb{R}$, not all zero. Considering the fourth coordinates of these vectors, we must then have $4c_1 + 2c_2 - 0$, or $c_2 = -2c_1$. Similarly, the first and second coordinates imply

$$c_3t = -c_2 = -2c_1,$$

so we would need to have $c_1 = c_2 = 0$. This would then force $c_3 \neq 0$, and hence $t = 0$—but even in this case, the third coordinate gives $2c_3 = 0$, a contradiction. Thus, no such linear dependence can exist for any t.

(4) Three friends go to the space needle. In geocentric coordinates, Emmy and Johann stand at positions $\mathbf{x}_1 = (2, 0, 0)$, $\mathbf{x}_2 = (1, 1, 0)$, respectively, and stare in the direction of vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 1, 2)$, respectively, towards Olga on the observation deck (see coverpage for an illustration.)

(a) (4pts) Model this problem with a system of 3 equations in 2 unknowns. We want to find two unknown scalars $c_1, c_2 \in \mathbb{R}$ such that $\mathbf{x}_1 + c_1\mathbf{v}_1 = \mathbf{x}_2 + c_2\mathbf{v}_2$. In standard form, this 3×2 linear system can be written as

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

(b) (4pts) Calculate Olga's position vector $\mathbf{x}_3$. First, we solve the system by row-reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ 3 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

hence $c_1 = 2$ and $c_2 = 3$. Olga's position equals

$$\mathbf{x}_3 = \mathbf{x}_1 + c_1\mathbf{v}_1 = \mathbf{x}_2 + c_2\mathbf{v}_2 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

(c) (2pts) Let A be the 3×2 coefficient matrix of the system from part a, and consider the associated linear transformation. Is T_A 1-1? Explain. Yes—from the RREF of A,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we see that there every column of A is a pivot column, and hence by the "unifying theorem" T_A is 1-1.

(5) (a) (4pts) Write down the matrix representation of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that sends a point $(x, y) \in \mathbb{R}^2$ to the closest point on the x-axis. The transformation in question is the projection onto the x-axis, whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) (2pts) Is the linear transformation from 5a) onto? Explain. No—the codomain of T is all of $\mathbb{R}^2$, but the range of T just the x-axis.

(c) (4pts) Write down the matrix representation of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects a vector across the y-axis, then rotates it 270° counterclockwise around the origin. The reflection sends e_1 to e_1 and e_2 to e_2, so its representing matrix is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The rotation sends e_1 to $-e_1$ and e_2 to e_1, so its representing matrix is

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, for the composite transformation, we obtain

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This could also be computed without matrix multiplication, by thinking about where the composite transformation sends the standard basis. Note additionally that $AB \neq BA$.